ON THE EVALUATION AND THE STUDY OF PARTICULAR SOLUTIONS OF THE GENERALIZED HILL EQUATION

(O VYCHISLENII I ISSLEDOVANII CHASTNYKH RESHENII OBOBSHCHENNOGO URAVNENIIA KHILLA)

PMM Vol.26, No.3, 1962, pp. 455-465

G.F. FILIMONOV
(Moscow)

(Received February 23, 1960)

The expansion and evaluation of Fourier coefficients for particular solutions of the generalized Hill equation are obtained by means of the direct solution of an infinite system of linear algebraic equations; convergence conditions are found. The practical applicability of different forms of particular solutions and different methods of writing the infinite system are considered. A closed expression containing only one unknown μ is derived for the characteristic equation.

1. One of the most prevalent methods of solving the generalized Hill equation

$$\frac{d^2}{d\tau^2} y(\tau) + \Psi(\tau) y(\tau) = 0, \qquad \Psi(\tau) = \sum_{m=-\infty}^{\infty} \theta_m \cos(2m\tau + \varepsilon_m) \tag{1.1}$$

is the representation of its particular solutions $y_k = y_k(\tau)$ $(k = 1, 2)$ in the form of expansions in all possible products of the parameters θ_m.

These solutions are obtained by the method of successive approximations, applied directly to Equation (1.1). However, in these methods the construction of the successive approximations by the usual means [1-3] generally do not allow us to investigate the convergence of the series obtained for $y_k(\tau)$. Below, the method of successive approximations is applied directly to a system of infinite algebraic equations, to which Equation (1.1) can be reduced, to compute the expansions for $y_k(\tau)$; in connection with this, convergence criteria are determined for the series obtained for $y_k(\tau)$.

2. Let us investigate infinite systems of equations. To compute $y_k(\tau)$ we must take the following expressions as given.

The first is Floquet's formula [1]

$$y_k(\tau) = e^{\mu_k \tau}\Phi_k(\tau) \qquad (k = 1, 2) \tag{2.1}$$

The second is a linear combination of Formulas (2.1)

$$y_1(\tau) = \Phi_1(\tau)\cos\nu\tau - \Phi_2(\tau)\sin\nu\tau, \quad y_2(\tau) = \Phi_1(\tau)\sin\nu\tau + \Phi_2(\tau)\cos\nu\tau \tag{2.2}$$

by setting $\mu_1 = -\mu_2 = i\nu$ in (2.1).

In (2.1), (2.2) the periodic functions $\Phi_k(\tau)$ can be represented in the form

$$\Phi_k(\tau) = C_k + \sum_{q=-\infty}^{\infty} \Delta(q; -n/2)\, A_{kq} \sin[(n+2q)\tau + \varphi_{kq}] \tag{2.3}$$

or

$$\Phi_k(\tau) = C_k + \sum_{q=-\infty}^{\infty} \Delta(q, -n/2)\, [c_{kq}\sin(n+2q)\tau + s_{kq}\cos(n+2q)\tau]$$

$$(c_{kq} = A_{kq}\cos\varphi_{kq}, \quad s_{kq} = A_{kq}\sin\varphi_{kq}) \tag{2.4}$$

Here n is an integer designating the regions of stability and instability of the solutions of Equation (1.1); the symbol is introduced

$$\Delta(q; r, s, \ldots, t) = (1-\delta_{qr})(1-\delta_{qs})\ldots(1-\delta_{qt}) \qquad \left(\begin{matrix}\delta_{\lambda\nu} = 0 \text{ when } \lambda \neq \nu \\ \delta_{\lambda\nu} = 1 \text{ when } \lambda = \nu\end{matrix}\right) \tag{2.5}$$

In (1.1), (2.3), (2.4), in order to write the Fourier series we shall assume that

$$\theta_{-m} = \theta_m, \qquad \varepsilon_{-m} = -\varepsilon_m, \qquad A_{k,\,-n-q} = A_{kq}, \qquad \varphi_{k,\,-n-q} = \pi - \varphi_{kq} \tag{2.6}$$

Thus, the unknowns will be the quantities A_{kq}, φ_{kq}, c_{kq}, s_{kq}, whose indices satisfy the inequality

$$n + 2q > 0 \tag{2.7}$$

Combining (2.3) (or (2.4)) with (2.1) (or (2.2)), we can obtain different expressions for $y_k(\tau)$. To every one of them there corresponds its own infinite system of algebraic equations*.

---

* The representation

$$\Phi_k(\tau) = C_k + \sum_{q=-\infty}^{\infty} \Delta(q; -n/2)\, [b_{kq}^{(+)} e^{i(n+2q)\tau} + b_{kq}^{(-)} e^{-i(n+2q)\tau}]$$

is not investigated: it has no particular advantage.

Direct substitution into (1.1) shows that when the functions $\Phi_k(\tau)$ are determined by (2.3), Equation (1.1) reduces to the system

$$I_q A_q = \sum_{m=-\infty}^{\infty} \Delta(m;\ q,\ -n-q)\,\zeta_{qm}(\varphi_m - \varphi_q + \pi/2)\,A_m \tag{2.8}$$

$$\tan(\varphi_q - \psi_q) = \frac{\sum\limits_{m=-\infty}^{\infty} \Delta(m;\ q,\ -n-q)\,[\zeta_{qm}(\varphi_m - \psi_q) + r_q\zeta_{qm}(\varphi_m - \psi_q + \pi/2)]\,A_m}{\sum\limits_{m=-\infty}^{\infty} \Delta(m;\ q,\ -n-q)\,[\zeta_{qm}(\varphi_m - \psi_q + \pi/2) - r_q\zeta_{qm}(\varphi_m - \psi_q)]\,A_m} \tag{2.9}$$

$$C = -\delta_{n,\,2N} \sum_{m=-\infty}^{\infty} \Delta(m;\ -n/2)\,A_m \frac{\theta_{N+m}}{\theta_0 + \mu^2} \sin(\varphi_m + \varepsilon_{N+m}) \tag{2.10}$$

and when the functions $\Phi_k(\tau)$ are determined by (2.4), to the system

$$\begin{aligned} L_q s_q &= \sum_{m=-\infty}^{\infty} \Delta(m;\ q,\ -n-q)\,[\alpha_{qm}s_m + \beta_{qm}c_m] \\ L_q c_q &= \sum_{m=-\infty}^{\infty} \Delta(m;\ q,\ -n-q)\,[\gamma_{qm}s_m + \varkappa_{qm}c_m] \end{aligned} \tag{2.11}$$

$$C = -\delta_{n,\,2N} \sum_{m=-\infty}^{\infty} \Delta(m;\ -n/2)\frac{\theta_{N+m}}{\theta_0 - \nu^2}\,[s_m \cos\varepsilon_{N+m} + c_m \sin\varepsilon_{N+m}] \tag{2.12}$$

In (2.8) to (2.12) the index k has been dropped from A_{kq}, φ_{kq}, c_{kq}, s_{kq}, C_k; the index N stands for

$$N = \frac{1}{4}[2n - 1 + (-1)^n] \tag{2.13}$$

The coefficients $\zeta_{qm}(x)$, r_q, ψ_q, I_q, α_{qm}, β_{qm}, γ_{qm}, κ_{qm}, L_q are derived below for every specific case.

Systems (2.8) and (2.11) are of like nature. Therefore, the unknowns A_q and c_q, s_q may be determined to within an arbitrary factor; usually we assume one of the amplitudes to be

$$A_{p_0} = 1 \tag{2.14}$$

The index p_0 is conveniently chosen from the condition $|I_{p_0}| < |I_p|$ or $|L_{p_0}| < |L_p|$, if $p_0 \neq p$. In the majority of cases these conditions are satisfied by $p_0 = 0$ if $n \neq 0$ and by $p_0 = 1$ if $n = 0$.

Condition (2.14) selects two equations each from the systems (2.8) to (2.10) and (2.11) to (2.12), corresponding to $q = p_0$. They play the role of characteristic equations and serve to determine μ and φ_0.

In the process of computing A_q, φ_q (or c_q, s_q) we should consider that the characteristic equations have been eliminated from systems (2.8) to (2.10) and (2.11) to (2.12), i.e. $q \neq p_0$, $q \neq -n-p_0$.

The terms of the sums with indices $m = p_0$ and $m = -n-p_0$ entering on the right-hand side of (2.8), (2.11), must be isolated, since they will be nonhomogeneous terms of the equations.

System (2.8) will be regular if

$$1 > \sum_{m=-\infty}^{\infty} \Delta(m;\ p_0, q, -n-p_0, -n-q) \left| \frac{\zeta_{qm}(\varphi_m - \varphi_q + \pi/2)}{I_m} \right| \tag{2.15}$$

System (2.11) will be regular if

$$\begin{aligned} 1 &> \sum_{m=-\infty}^{\infty} \Delta(m;\ p_0, q, -n-p_0, -n-q)\ \frac{|\alpha_{qm}| + |\beta_{qm}|}{|L_m|} \\ 1 &> \sum_{m=-\infty}^{\infty} \Delta(m; p_0, q, -n-p_0, -n-q)\,\frac{|\gamma_{qm}| + |\varkappa_{qm}|}{|L_m|} \end{aligned} \tag{2.16}$$

If conditions (2.15), (2.16) are fulfilled, every expansion for A_q and c_q, s_q, obtained by solving (2.8) and, respectively, (2.11) by the method of successive approximations, will converge absolutely [4].

Inequalities (2.15) and (2.16) are not equally strong; for example, in the case $\theta_m \to 0$, $m \neq 0$, it can be shown that (2.15) is weaker than (2.16).

Inequalities (2.15) and (2.16) give the absolute criteria for convergence. It can be proved that if (2.15) and (2.16) are fulfilled simultaneously then the correct convergence criterion is the weaker one of (2.15), (2.16).

Below (see (5.13)), we shall find the necessary convergence criteria from the solutions of systems (2.8), (2.9) for the case of θ_m decreasing by a geometrical progression of growth $|m|$.

Comparing (5.13) with (2.15) we can convince ourselves that these conditions are not greatly different from each other.

The suggested methods for computing $y_k(\tau)$ are related to Whittaker's method (see [1, p.253]). The solution (2.2), (2.4) occupies a special place in them. It includes the closed characteristic equation and allows an independent computation of the characteristic number ν and of the Fourier coefficient c_{kq}, s_{kq}.

3. Let us study the solution (2.2), (2.4). In this case we obtain the system of equations (2.11) to (2.12) whose coefficients equal

$$\alpha_{qm} = \varkappa_{qm} = \zeta_{qm}(\pi/2), \qquad \beta_{qm} = -\gamma_{qm} = \zeta_{qm}(0) \tag{3.1}$$

Here

$$\zeta_{qm}(x) = \left(1 + \frac{n+2q}{M_q}\frac{M_m}{n+2m}\right)\Delta(m;\, q,\, -n/2)\,[\theta_{q-m}\sin(x + \varepsilon_{m-q}) - $$
$$- \lambda_{qm}\sin(x + \varepsilon_{N+m} - \varepsilon_{N+q})]$$
$$- \Delta(m;\, -n/2)\sum_{p=-\infty}^{\infty}\frac{\Delta(p;\, q, m, -n/2)}{n+2p}\,\frac{n+2q}{M_q}\,[\theta_{p-m}\theta_{q-p}\sin(x + \varepsilon_{m-p} + \varepsilon_{p-q}) +$$
$$+ \lambda_{pp}\lambda_{qm}\sin(x + \varepsilon_{N+m} - \varepsilon_{N+q}) - \theta_{p-m}\lambda_{pq}\sin(x + \varepsilon_{m-p} + \varepsilon_{N+p} - \varepsilon_{N+q}) -$$
$$- \theta_{q-p}\lambda_{pm}\sin(x + \varepsilon_{p-q} + \varepsilon_{N+m} - \varepsilon_{N+p})]$$
$$M_q = (n+2q)^2 - \theta_0 + \nu^2 + \lambda_{qq}, \qquad \lambda_{qm} = \delta_{n,\,2N}\frac{\theta_{N+m}\theta_{N+q}}{\theta_0 - \nu^2} \tag{3.2}$$
$$L_q = M_q - 4\nu^2\frac{(n+2q)^2}{M_q} - \zeta_{qq}(\pi/2)$$

In solution (2.2) only one amplitude, namely A_{1p_0}, can be assigned arbitrarily. Therefore, by selecting A_{1p_0} according to (2.14), the amplitude A_{2p_0} should be computed from the equality

$$2\nu(n+2q)\,A_{2q}\sin(x + \varphi_{2q}) = M_q A_{1q}\cos(x + \varphi_{1q}) - \tag{3.3}$$
$$- \sum_{m=-\infty}^{\infty}\Delta(m;\, q,\, -n/2)\,A_{1m}\,[\theta_{q-m}\cos(x + \varphi_{1m} + \varepsilon_{m-q}) -$$
$$- \lambda_{qm}\cos(x + \varphi_{1m} + \varepsilon_{N+m} - \varepsilon_{N+q})]$$

arising from (1.1).

The phase x in Equalities (3.2), (3.3) is arbitrary. The solutions of system (2.11), (3.1), (3.2) can be found by means of successively eliminating the unknowns c_q, s_q from Equations (2.11). As a result we obtain the expression

$$L_q^{(\infty)}c_q = \Delta(q;\, p_0)\,(\gamma_{qp_0}^{(\infty)}s_{p_0} + \varkappa_{qp_0}^{(\infty)}c_{p_0})$$
$$L_q^{(\infty)}s_q = \Delta(q;\, p_0)\,(\alpha_{qp_0}^{(\infty)}s_{p_0} + \beta_{qp_0}^{(\infty)}c_{p_0}) \tag{3.4}$$

whose coefficients are determined by the recurrence formulas

$$\begin{aligned}
\alpha_{qm}^{(k+1)} &= \alpha_{qm}^{(k)} + \frac{\Delta(p_k;\, q)}{L_{p_k}^{(k)}} [\alpha_{qp_k}^{(k)} \alpha_{p_k m}^{(k)} + \beta_{qp_k}^{(k)} \gamma_{p_k m}^{(k)}] \\
\beta_{qm}^{(k+1)} &= \beta_{qm}^{(k)} + \frac{\Delta(p_k;\, q)}{L_{p_k}^{(k)}} [\alpha_{qp_k}^{(k)} \beta_{p_k m}^{(k)} + \beta_{qp_k}^{(k)} \varkappa_{p_k m}^{(k)}] \\
\gamma_{qm}^{(k+1)} &= \gamma_{qm}^{(k)} + \frac{\Delta(p_k;\, q)}{L_{p_k}^{(k)}} [\gamma_{qp_k}^{(k)} \alpha_{p_k m}^{(k)} + \varkappa_{qp_k}^{(k)} \gamma_{p_k m}^{(k)}] \qquad (3.5) \\
\varkappa_{qm}^{(k+1)} &= \varkappa_{qm}^{(k)} + \frac{\Delta(p_k;\, q)}{L_{p_k}^{(k)}} [\varkappa_{qp_k}^{(k)} \varkappa_{p_k m}^{(k)} + \gamma_{qp_k}^{(k)} \beta_{p_k m}^{(k)}]
\end{aligned}$$

$$L_q^{(k+1)} = L_q^{(k)} - \Delta(q;\, p_1, p_2, \ldots, p_k)[\alpha_{qq}^{(k+1)} - \alpha_{qq}^{(k)}],\ L_q^{(0)} = L_q$$

$$\alpha_{qm}^{(0)} = \zeta_{qm}(\pi/2) + \zeta_{q,\,-n-m}(\pi/2), \qquad \gamma_{qm}^{(0)} = -\zeta_{qm}(0) - \zeta_{q,\,-n-m}(0)$$

$$\beta_{qm}^{(0)} = \zeta_{qm}(0) - \zeta_{q,\,-n-m}(0), \qquad \varkappa_{qm}^{(0)} = \zeta_{qm}(\pi/2) - \zeta_{q,\,-n-m}(\pi/2)$$

In (3.4), (3.5) the indices $p_k \neq p_j$ if $k \neq j$; the set p_k coincides with the set of numbers $N+1-n$, $N+2-n$, ...

The sequence of changes in p_k, depending on the index k, is arbitrary and is determined by the order in which c_q, s_q are eliminated from system (2.11): c_{p_1}, s_{p_1} are eliminated first, then c_{p_2}, s_{p_2}, etc.

In practice it is more convenient to start with the elimination of the c_q, s_q with the largest amplitudes.

In Equalities (2.11) and (3.4), to the number $q = p_0$ there correspond two equations which are the last in the sequential elimination of c_q, s_q. These equations play the role of characteristic equations.

From (3.4) it is evident that when $q = p_0$ the right-hand sides of (3.4) vanishes. If $A_{p_0} \neq 0$, the quantities c_{p_0}, s_{p_0} cannot vanish simultaneously (see (2.4)). Therefore, phase φ_{p_0} is arbitrary*, and the characteristic equation has the form

$$L_{p_0}^{(\infty)} = L_{p_0} - \sum_{p_k = N+1-n}^{\infty} \Delta(p_k;\, p_0) \frac{1}{L_{p_k}^{(k)}} [\alpha_{p_0 p_k}^{(k)} \alpha_{p_k p_0}^{(k)} + \beta_{p_0 p_k}^{(k)} \gamma_{p_k p_0}^{(k)}] = 0 \quad (3.6)$$

---

* φ_{1p_0} is chosen arbitrarily; phase φ_{2p_0} (as also A_{2p_0}) should be computed from (3.3).

There remains only one unknown ν. Therefore, the solution of (3.6) can be sought for independently of the calculation of c_q, s_q. As an example, below, approximate expressions for the roots of (3.6) are derived for the case

$$|\sqrt{\theta_0} - m| \gg \theta_1, \qquad |m| = 0, 1, \ldots$$

In the zeroth approximation

$$\nu_0 = n + 2p_0 - \left(\theta_0 + \frac{\delta_{n,\,2N}\theta_1^2}{(n + 2p_0)(n + 2p_0 - 2\sqrt{\theta_0})}\right)^{1/2} \tag{3.7}$$

In the first approximation

$$\begin{aligned} \nu^2 &= \nu_0^2 - \frac{\theta_1^2}{2\sqrt{\theta_0}(1 + \sqrt{\theta_0})}, \quad n = 1, \quad p_0 = 0 \\ \nu^2 &= \nu_0^2 - \nu_0 \frac{\theta_1^2}{4\sqrt{\theta_0}(1 + \sqrt{\theta_0})}, \quad n = 2, \quad p_0 = 0 \end{aligned} \tag{3.8}$$

It is known [1] that the Hill determinant (the determinant of system (2.11)) has an infinite number of roots; at the same time (1.1) cannot have more than two characteristic exponents. Formula (3.7) shows that whether one or the other of the roots of the Hill determinant appears as a characteristic exponent or not, depends upon the method of solving (1.1).

In particular, for real τ and $\Psi(\tau)$ the characteristic numbers μ_k ($k = 1, 2$) of solutions (2.1), are either real or pure imaginary. The proof of this assertion is simple. For real τ and $\Psi(\tau)$ both $y_1(\tau)$ and $y_1^*(\tau)$ will be solutions of (1.1). Therefore, μ_1 and μ_1^* will be the characteristic exponents of Equation (1.1) simultaneously, and either $\mu_1^* = \mu_1$ or $\mu_1^* = \mu_2 = -\mu_1$.

The application of Equation (3.6) requires the evaluation of $\zeta_{qm}(x)$ from the complicated Expression (3.2). In those cases where the series for $\Psi(\tau)$ in (1.1) contains a finite number of terms, Expression (3.2) simplifies. Otherwise, the sums in (3.2) can be represented by use of known trigonometrical series (see, for example, [5, (1.445)]) in the form of definite integrals the evaluation of which may be simpler than direct summation.

If, however, simple expressions cannot be found for the coefficients of (3.2), the application of expansions (3.4) to (3.5) for computing c_q, s_q becomes inappropriate.

In this case, by computing ν from the characteristic Equation (3.6), other solutions which lead to coefficients simpler than (3.1), (3.2) can be used instead of (2.2), (2.4) for determining $y_k(\tau)$.

4. As such a solution let us study (2.1), (2.4). In this case the coefficients α_{qm}, β_{qm}, γ_{qm}, κ_{qm} can be represented as combinations of two functions

$$
\begin{aligned}
\alpha_{qm} &= \Delta(m;\ -n/2)\,[\xi_{qm}(\pi/2) + \sigma_{qm}(\pi/2)] \\
\beta_{qm} &= \Delta(m;\ -n/2)\,[-\xi_{qm}(0) - \sigma_{qm}(0)] \\
\gamma_{qm} &= \Delta(m;\ -n/2)\,[\xi_{qm}(0) - \sigma_{qm}(0)] \\
\varkappa_{qm} &= \Delta(m;\ -n/2)\,[\xi_{qm}(\pi/2) - \sigma_{qm}(\pi/2)]
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
\xi_{qm}(x) = {} & (M_q + \lambda_{qq})\,[\theta_{q-m}\sin(x + \varepsilon_{q-m}) - \lambda_{qm}\sin(x + \varepsilon_{N+q} - \varepsilon_{N+m})] - \\
& - 2\mu\,(n+2q)\,[\theta_{q-m}\cos(x + \varepsilon_{q-m}) - \lambda_{qm}\cos(x + \varepsilon_{N+q} - \varepsilon_{N+m})]
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{qm}(x) = {} & \theta_{n+2q}\,[\theta_{q-m}\sin(x + \varepsilon_{q-m} - \varepsilon_{n+2q}) - \lambda_{qm}\sin(x + \varepsilon_{N+q} - \varepsilon_{N+m} - \varepsilon_{N+2q})] - \\
& - \lambda_{qq}\,[\theta_{q-m}\sin(x + \varepsilon_{q-m} - 2\varepsilon_{N+q}) - \lambda_{qm}\sin(x - \varepsilon_{N+q} - \varepsilon_{N+m})]
\end{aligned}
\tag{4.2}
$$

$$
M_q = (n+2q)^2 - \theta_0 - \mu^2, \qquad \lambda_{qm} = \delta_{n,\,2N}\,\frac{\theta_{N+q}\theta_{N+m}}{\theta_0 + \mu^2}
$$

$$
L_q = M_q^{\,2} + 4\mu^2(n+2q)^2 + 2\lambda_{qq}\,[M_q + \theta_{n+2q}\cos(\varepsilon_{N+2q} - 2\varepsilon_{N+q})] - \theta_{n+2q}^{\,2}
$$

In order to find solutions of Equations (2.11), (4.1), (4.2), let us write system (2.11) in the form*

$$
a_q = g_q + \sum_{m=-2N}^{\infty} \Delta(m;\ 0,\ 1)\, z_{qm} a_m
\tag{4.3}
$$

$$
\begin{aligned}
a_{2q} &= L_q c_q, & L_m z_{2q,\,2m} &= \Delta(m;\ q)\,[\varkappa_{qm} - \varkappa_{q,\,-n-m}] \\
a_{2q+1} &= L_q s_q, & L_m z_{2q,\,2m+1} &= \Delta(m;\ q)\,[\gamma_{qm} + \gamma_{q,\,-n-m}] \\
g_q &= a_0 z_{q0} + a_1 z_{q1}, & L_m z_{2q+1,\,2m} &= \Delta(m;\ q)\,[\beta_{qm} - \beta_{q,\,-n-m}] \\
& & L_m z_{2q+1,\,2m+1} &= \Delta(m;\ q)\,[\alpha_{qm} + \alpha_{q,\,-n-m}]
\end{aligned}
$$

To compute a_m by means of (4.3), different variations of the method of successive approximations can be suggested. When the regularity conditions (2.16) are fulfilled, they all lead to one and the same value of a_q (see [4]). However, each of them turns out to be the most suitable only for a specific rule of variation of the numbers z_{qm} with increasing indices q, m.

---

* Here for brevity we assume $p_0 = 0$.

As an example let us investigate two elementary cases. Let θ_m be a slowly varying sequence of numbers. In this case z_{qm} equals $\theta_{q-m} L_m^{-1}$ in magnitude; the quantity a_q is appropriately sought for in the form

$$z_{qm} = tU_{qm}, \qquad a_q = \sum_{k=0}^{\infty} a_{qk} t^k \tag{4.4}$$

Substituting (4.4) into (4.3) and equating the coefficients of like powers of t, we find

$$a_{qk} = g_q \delta_{k0} + \Delta(k;\ 0) \sum_{p_1=-2N}^{\infty} \Delta(p_1;\ 0,\ 1)\, U_{qp_1} \sum_{p_2=-2N}^{\infty} \Delta(p_2;\ 0,\ 1)\, U_{p_1 p_2} \cdots$$
$$\cdots \sum_{p_k=-2N}^{\infty} \Delta(p_k;\ 0,\ 1)\, U_{p_{k-1} p_k} g_{p_k} \tag{4.5}$$

Formulas (4.4), (4.5) give the final expression a_q if we set $t = 1$ for the auxiliary quantity t which occurs in them.

Let us now assume that in magnitude $\theta_m \sim x^{|m|}$ $(|m| = 0,\ 1,\ 2,\ \ldots)$.

In this case $z_{qm} \sim x^{|Q-M|}$ in magnitude, and a_q is appropriately sought for in the form*

$$z_{qm} = t^{|Q-M|} V_{qm}, \qquad g_q = t^{|Q|} G_q, \qquad a_q = \sum_{k=0}^{\infty} b_{qk} t^{k+|Q|} \tag{4.6}$$

Substituting (4.6) into (4.5) and equating the coefficients of t, we find

$$b_{qk} = G_q \delta_{k0} + \sum_{m=0}^{2Q+1} \Delta(m;\ 0,\ 1)\, V_{qm} b_{mk} + \tag{4.7}$$
$$+ \left\{ \sum_{m=-2N}^{2QD(-Q)} \delta_{r,\, k+2[M-QD(-Q)]} + \sum_{m=2(Q+1)D(Q)}^{\infty} \delta_{r,\, k-2[M-QD(Q)]} \right\} \Delta(m;\ 0,\ 1)\, V_{qm} b_{mr}$$

$$D(Q) = \begin{cases} 1 & (Q > 0) \\ 0 & (Q < 0) \end{cases}$$

Equalities (4.7) represents a set of recurrence relations which allow us to compute the quantities b_{qk} by their "lowest" approximations b_{mr}

---

* The numbers Q and M are computed by means of Formulas (2.10) in which q and m, respectively, should be substituted for n.

($r < k$) and by the approximations of the same order of the "previous" amplitudes $b_{mk}(|m| < |q|)$. It is not difficult to apply (4.7) in practice. Equalities (4.6), (4.7) give the final expression for a_q if we set $t = 1$ in them.

It is not difficult to see that if condition (2.14) is fulfilled the coefficients z_{qm} depend on θ_m, ε_m, μ, and g_q on θ_m, ε_m, μ, φ_0. If μ is computed by use of Equation (3.6), then expansions (4.4) to (4.7) will depend only on the unknown φ_0. The characteristic equation of system (2.11), (4.1), (4.2) can be presented in the form of two equations

$$\begin{gathered}\theta_0 - n^2 + \mu^2 = \theta_n \cos(2\varphi_0 + \varepsilon_n) + 2\lambda_{00}\sin^2(\varphi_0 + \varepsilon_N) - \\ - \sum_{m=-\infty}^{\infty} \Delta(m; 0, -n/2, -n) A_m [\theta_m \cos(\varphi_m - \varphi_0 + \varepsilon_m) - \lambda_{0m}\cos(\varphi_m - \varphi_0 + \varepsilon_{N+m} - \varepsilon_N)] \\ 2n\mu = -\theta_n \sin(2\varphi_0 + \varepsilon_n) + \lambda_{00}\sin 2(\varphi_0 + \varepsilon_N) - \\ - \sum_{m=-\infty}^{\infty} \Delta(m; 0, -n/2, -n) A_m [\theta_m \sin(\varphi_m - \varphi_0 + \varepsilon_m) - \lambda_{0m}\sin(\varphi_m - \varphi_0 + \varepsilon_{N+m} - \varepsilon_N)]\end{gathered} \tag{4.8}$$

By substituting here (4.4) to (4.7) we find c_0, s_0, and then we compute $y_k(\tau)$.

5. Let us study the solutions (2.1), (2.3) and (2.2), (2.3) in order to find approximate expressions for A_q and φ_q. Here, Equation (1.1) can be transformed to the system (2.8) to (2.10) whose coefficients equal

$$\begin{gathered}\zeta_{qm}(x) = \Delta(m; -n/2)[\theta_{q-m}\sin(x + \varepsilon_{m-q}) - \lambda^*_{qm}\sin(x + \varepsilon_{N+m} - \varepsilon_{N+q})] \\ I_q = (n + 2q)^2 - \theta_0 - \mu^2 + \lambda_{qq}[1 - \cos 2(\varphi_q + \varepsilon_{N+q})] + \theta_{n+2q}\cos(2\varphi_q + \varepsilon_{n+2q}) \\ \lambda^*_{qm} = \delta_{n,2N}\frac{\theta_{N+m}\theta_{N+q}}{\theta_0 + \mu^2}, \qquad r_q = \frac{2\mu(n + 2q) + \zeta_{-n-q,\,q}(2\varphi_q)}{I_q}\end{gathered} \tag{5.1}$$

in the case of solution (2.1), (2.3). For solution (2.2), (2.3)

$$r_q = 0, \qquad I_q = L_q \tag{5.2}$$

and L_q, $\zeta_{qm}(x)$ are determined by means of (3.2).

The coefficients of system (2.8) to (2.10) depend on the phases φ_m. Therefore it is inconvenient to make use of Expression (2.3) for an accurate computation of $y_k(\tau)$. However, for real values of the phases φ_m, when it is possible to dominate by unity all the trigonometrical functions occurring in (2.8) to (2.10), Equations (2.8) to (2.10) allow us to obtain approximate expressions for A_q and φ_q.

Let us investigate the case for real θ_m, ε_m. In this case, the phases

φ_m of solution (2.1), (2.3) will be real in the instability regions, and the phases φ_m of solution (2.2), (2.3) in the stability regions. Therefore, to evaluate relations for the stability of the solution of (1.1) we must use (2.2), (2.3), and for instability, (2.1), (2.3).

To a large extent the nature of the variations of A_q with the increase of $|q|$, depends on the coefficients θ_m. Therefore, below, we study the approximate expressions for A_q in two special cases.

If θ_m, $|m| = 1, 2, \ldots$, forms a slowly varying sequence of numbers, then A_q is conveniently sought for in a form analogous to (4.4). Let us introduce the notation

$$z_{qm} = \Delta(m;\ q)\,[\delta_{m0} + I_m^{-1}\Delta(m;\ 0)]\,[\zeta_{qm}(\varphi_m - \varphi_q + \pi/2) - \\ - \zeta_{q,\,-n-q}(-\varphi_m - \varphi_q + \pi/2)] \tag{5.3}$$

Let us set*

$$z_{qm} = tU_{qm}, \qquad A_q I_q = A_0 \sum_{k=0}^{\infty} a_{qk} t^{k+1} \tag{5.4}$$

By using (2.8) we find

$$a_{qk} = \delta_{k0}U_{q0} + \Delta(k;\ 0) \sum_{p_1=-N}^{\infty} \Delta(p_1;\ 0)\,U_{qp_1} \sum_{p_2=-N}^{\infty} \Delta(p_2;\ 0)\,U_{p_1p_2} \cdots \\ \cdots \sum_{p_k=-N}^{\infty} \Delta(p_k;\ 0)\,U_{p_{k-1}p_k} U_{p_k0} \tag{5.5}$$

For A_q we obtain

$$|I_q A_q| \leqslant |A_0| \left[|z_{q0}| + \frac{S^{(0)}}{1-S} \right]$$

$$S \geqslant \sum_{m=-\infty}^{\infty} \Delta(m;\ 0,\ q,\ -n,\ -n-q) \left| \frac{\zeta_{qm}(\varphi_m - \varphi_q + \pi/2)}{I_m} \right| \tag{5.6}$$

$$S^{(0)} \geqslant \left| \sum_{m=-N}^{\infty} z_{qm} z_{m0} \right|$$

The quantity S is the majorant of the right-hand sides of (2.15).

Inequality (5.6) takes a simple form if we assume

---

* Let us recall that in solution (2.2), (2.3) the arbitrary A_{10}, φ_{10}, A_{20}, φ_{20} must be calculated by using (3.3).

$$|\zeta_{qm}(\varphi_m - \varphi_q + \pi/2) - \zeta_{q,\,-n-m}(-\varphi_m - \varphi_q + \pi/2)| < t \tag{5.7}$$

The interval of values of the quantity t, within the limits of which inequalities (5.6) have meaning, is determined with the aid of (2.15).

If θ_m, $|m| = 1, 2, \ldots$, forms a geometric progression, then A_q is conveniently sought for in a form analogous to (4.6).

Let us set

$$z_{qm} = x^{|q-m|}V_{qm}, \qquad I_q A_q = A_0 \sum_{k=0}^{\infty} b_{qk} x^{k+|q|} \tag{5.8}$$

Substituting (5.8) into (2.8) and equating the coefficients of powers of x, we find

$$b_{qk} = \delta_{k0} V_{q0} + \sum_{m=0}^{q} \Delta(m;\,0) V_{qm} b_{mk} + + \left\{ \sum_{m=-N}^{qD(-q)-1} \delta_{r,\,k+2[m-qD(-q)]} + \sum_{m=qD(q)+1}^{\infty} \delta_{r,\,k-2[m-qD(q)]} \right\} V_{qm} b_{mr} \tag{5.9}$$

If

$$|\zeta_{qm}(\varphi_m - \varphi_q + \pi/2) - \zeta_{q,\,-n-m}(-\varphi_m - \varphi_q + \pi/2)| \leqslant t x^{|q-m|} \tag{5.10}$$

then Equalities (5.9) allow us to derive the approximate expression

$$|b_{q0}| \lessapprox t \left(1 + \frac{t}{|I_{1\,\mathrm{sign}\,q}|}\right)\left(1 + \frac{t}{|I_{2\,\mathrm{sign}\,q}|}\right)\cdots\left(1 + \frac{t}{|I_q|}\right) \tag{5.11}$$

With known accuracy I_m in (5.11) can be replaced by $I_m \approx 4m^2$; here, as $q \to \infty$, the right-hand side of (5.11) is transformed to a known infinite product, and

$$|b_{q0}| \lessapprox t \frac{\sinh(\pi\sqrt{t}/2)}{\pi\sqrt{t}/2} \tag{5.12}$$

The range of variation of t and x, within the limits of which inequalities (5.11), (5.12) have meaning, can be determined by substituting (5.10) into (2.15).

By analysing the subsequent terms of the series in (5.8) for the case of (5.10), we can find the necessary convergence criterion for the series in (5.8)

$$\sum_{m=1}^{\infty} \frac{tx}{|I_m|}\,\frac{tx}{|I_{m+1}|} < 1 \tag{5.13}$$

The expansions (5.8), (5.9) derived here for the amplitudes A_q are similar to the series derived in [1,2] for the solution (2.1), (2.3).

It is not difficult to convince ourselves that (5.8), (5.9) are their generalizations. To see this, (5.1) should be substituted into (5.9) and I_m^{-1} should be expanded into a series of products of the numbers θ_m. Then (5.9) will coincide with the corresponding results of [1,2]. It is necessary, however, to note that when computing (2.1), (2.3) in the stability regions corresponding to $n > 2$, such an expansion becomes invalid since it leads to a violation of condition (2.15). Therefore, the expansion of $y_k(\tau)$ for $n = 3$, derived in [2], trivially applies only in the instability regions.

Let us study the approximate solutions of Equation (2.9) for the case of (5.10) when all the products of the form $A_q \ldots A_r \theta_\omega \ldots \theta_\varepsilon$ can be regarded as quantities of the pth order of smallness if $|q| + \ldots + |r| + |\omega| + \ldots + |\varepsilon| = p$.

In accordance with this classification the numerators of the right-hand sides of (2.9) contain terms of $|q| + 2p$, $n + q + 2p$ $(p = 0, 1, \ldots)$ orders of smallness.

The quantities ψ_q occurring in (2.9) are arbitrary. In particular, we can assume $\psi_q = 0$. However, the quantities ψ_q can also be chosen such that the right-hand sides of (2.9) have small magnitudes of the first or more orders of smallness. In this case ψ_q will approximate the values of phases φ_q.

To derive the equations defining ψ_q let us equate to zero the sum of all terms of the qth order of smallness occurring in the numerators of the right-hand sides of (2.9). Then we obtain:

for the instability regions (for solution (2.1), (2.3))

$$\psi_0 = \varphi_0, \qquad \sum_{m=0}^{q} \Delta(m;\ q, -n/2)\, A_m \theta_{q-m} \sin(\varphi_m - \psi_q + \varepsilon_{m-q}) = 0 \tag{5.14}$$

for the stability regions (for solution (2.2), (2.3))

$$\sum_{r=0}^{q} \Delta(r;\ q, -n/2)\left(\frac{M_q}{n+2q} + \frac{M_r}{n+2r}\right) A_r \theta_{q-r} \sin(\varphi_r - \psi_q + \varepsilon_{r-q}) -$$

$$- \sum_{m,\,p} \Delta(p;\, m,\ q, -n/2)\, \Delta(m;\ q, -n/2)\, \frac{\theta_{q-p}\theta_{p-m}}{n+2p} A_m \sin(\varphi_m - \psi_q + \varepsilon_{m-p} + \varepsilon_{p-q}) = 0 \tag{5.15}$$

$$0 \leqslant m < p < q, \qquad q < p < m \leqslant 0 \qquad (\psi_0 = \varphi_0)$$

If in (5.14), (5.15) we replace φ_m by ψ_m then (5.14), (5.15) will also give the desired solutions for ψ_q. Equations (5.14), (5.15) have the character of recurrence relations. They permit a simple series solution.

6. Let us study the solutions of the ordinary Hill equation, when $\varepsilon_m = 0$. System (2.9), corresponding to (2.2), (2.3), when $\varepsilon_m = 0$ has the solutions

$$\varphi_q = 0, \qquad \varphi_q = \pi/2 \tag{6.1}$$

Therefore, expansions (5.4), (5.5) and (5.8), (5.9), applied to solution (2.2), (2.3), in the case of the ordinary Hill equation contain only the unknown ν and may be regarded as final. (According to (3.3) phase $\varphi_{2q} = 0,\ \pi/2$ if $\varphi_{1q} = \pi/2,\ 0$, respectively.)

To compute the periodic ($\mu = 0$) solutions of the ordinary Hill equation we can also use Expressions (2.1), (2.3). In this case (6.1) will also satisfy system (2.9) and, (5.4) to (5.5), (5.8) to (5.9) will give the final expressions for amplitudes A_q. The advantage of the latter method of computing $y_k(\tau)$ is that the coefficients (5.1) are considerably simpler than the coefficients (3.2).

7. Let us investigate the Mathieu equation. In this case

$$\varepsilon_m = 0, \qquad \theta_m = \theta_0\delta_{m0} + \theta_1\delta_{|m|,1} \tag{7.1}$$

The constant C is conveniently taken inside the summation sign in (2.3) with the help of the inequalities

$$C = \delta_{n,2N}A_{-N}, \qquad \delta_{n,2N}\varphi_{-N} = \pi/2 \tag{7.2}$$

The system (2.8), (2.9) with coefficients

$$\zeta_{qm}(x) = \theta_1\delta_{|m-q|,1}\sin x, \qquad r_q = \frac{2\mu(n+2q)}{I_q}, \qquad I_q = (n+2q)^2 - \theta_0 - \mu^2 \tag{7.3}$$

is obtained for A_q, φ_q.

The Mathieu equation is a special case of the ordinary Hill equation. However, the methods set forth in Section 6 for computing $y_k(\tau)$ are not always convenient for solving the Mathieu equation since they lead to very crude estimates of the Fourier coefficients of $y_k(\tau)$ and of the convergence conditions.

Therefore, to solve the Mathieu equation it is appropriate to use the Whittaker series introduced in [1, p.256] which converges faster than (5.4) does.

In this case the application of infinite equations allows us to derive an expression for the general term of the Whittaker series, to find a quantitative estimate for their regions of convergence, and, to obtain convenient estimates for the quantities A_q, φ_q.

By assuming

$$A_q = \delta_{q0} + \Delta(q;\,0)\, a_q \frac{\theta_1 \cos(\varphi_{1\cdot\mathrm{sign}q} - \varphi_0)}{I_{1\cdot\mathrm{sign}q}} \frac{\theta_1 \cos(\varphi_{2\cdot\mathrm{sign}q} - \varphi_{1\cdot\mathrm{sign}q})}{I_{2\cdot\mathrm{sign}\,q}} \cdots$$
$$\cdots \frac{\theta_1 \cos(\varphi_q - \varphi_{q-1\cdot\mathrm{sign}\,q})}{I_q} \qquad (7.4)$$

we find for a_q the system of equations

$$a_0 = 1, \qquad a_q = a_{q-1\cdot\mathrm{sign}\,q} + z_q a_{q+1\cdot\mathrm{sign}\,q}, \qquad z_q = \frac{\theta_1^2 \cos^2(\varphi_q - \varphi_{q+1\cdot\mathrm{sign}\,q})}{I_q I_{q+1\cdot\mathrm{sign}\,q}} \qquad (7.5)$$

System (7.5) decomposes naturally into two independent parts, corresponding to $q > 0$ and $q < 0$. By virtue of (2.8), only the system corresponding to $q > 0$ turns out to be infinite. We shall find its solution.

If in (7.5) we set $q > 0$, then we obtain an irregular system of equations to which the general methods stated in [4] do not apply. Nevertheless, the solution of (7.5) for $q > 0$ can be obtained with the aid of expansions of type (5.4)

$$z_q = tU_q, \qquad a_q = \sum_{k=0}^{\infty} a_{qk} t^k \qquad (7.6)$$

Substituting (7.6) into (7.5) we find

$$a_{qk} = \delta_{k0} + \Delta(k;\,0) \sum_{p_1=1}^{q} U_{p_1} \sum_{p_2=1}^{p_1+1} U_{p_2} \cdots \sum_{p_k=1}^{p_{k-1}+1} U_{p_k} \qquad (7.7)$$

It is not difficult to show that series (7.6), (7.7) converge absolutely if

$$1 > \left| \sum_{p=1}^{\infty} z_p \right| \qquad (7.8)$$

Here for a_q we obtain

$$|a_q| < (1 - S)^{-1}, \qquad S = \left| \sum_{p=1}^{\infty} z_p \right| \qquad (7.9)$$

Expansions (7.6), (7.7) represent a generalization of the Whittaker series. Inequality (7.8) gives the sufficient condition for their absolute convergence.

Let us consider the approximate solutions of system (2.9). By repeating the derivation of Equations (5.14) for ψ_q, we get

$$\psi_q = \psi_{q-1 \cdot \operatorname{sign} q} + \tan^{-1} \frac{2\mu (n + 2q)}{I_q}, \qquad \psi_0 = \varphi_0 \tag{7.10}$$

From (7.10) it is obvious that $\varphi_{q\pm 1} - \varphi_q$ is a small quantity tending to zero as $q \to \infty$. Therefore, in contrast to the general case (see Section 5), Expressions (2.1), (2.3) are suitable for the computation of both the unstable and the stable solutions of the Mathieu equation. From (7.9) we can also derive that

$$|\operatorname{Im} (\varphi_q - \varphi_0)| \approx {}^1\!/_2 \, |\mu| \ln |q (n + q)|$$

when $q \to \infty$ and μ has pure imaginary values.

8. One of the basic aims of the present paper is to obtain for the solutions of the generalized Hill equation simple expansions which allow direct practical application. Such desired expansions were obtained in Sections 3, 4, 6, depending only on the characteristic number μ.

If (3.6) allows a simple computation of ν, the use of series (3.4), (4.4) to (4.7), (5.4), (5.8) presents no difficulty.

If, however, it is not possible to solve (3.6) by simple means, then we must use other methods to compute $y_k(\tau)$. In particular, the method – suggested in [1] and applied in [2] – of the formal expansion of all quantities in terms of products of the numbers θ_m, can be useful. In this case conditions (2.15), (2.16), (5.13) and inequalities (5.6), (5.11) make it possible to find the accuracy of the method of successive approximations. When solving a truncated system of equations, inequalities (5.6), (5.11), (7.9) allow us to estimate the errors which arise from the trucnation of the infinite system.

In conclusion let us note that the results of Sections 4 to 7 are given for $p_0 = 0$ (see (2.14)). Therefore, we must be cautious when applying them to the case $n = 0$.

The author thanks the review editor K.G. Valeev for much help and valuable advice.

BIBLIOGRAPHY

1. Whittaker, E.G. and Watson, G.N., *Kurs sovremennogo analiza (Course in Modern Analysis)*, Chap. 2, GITTL, 1934.

2. Kayashi, T., *Vynuzhdennye kolebaniia v nelineinykh sistemakh (Forced Oscillations in Nonlinear Systems)*. IIL, 1957.

3. Courant, E. and Snyder, H., Theory of a synchrotron with strong focusing. *Ann. Phys.* Vol. 3, No. 1, 1958.

4. Kantorovich, L.V. and Krylov, V.I., *Priblizhennye metody vysshego analiza (Approximate Methods of Higher Analysis)*. GITTL, 1949.

5. Ryzhik, I.M. and Gradshtein, I.S., *Tablitsy integralov, riadov, summ i beskonechnykh proizvedenii (Tables of Integrals, Series, Sums and Infinite Products)*. GITTL, 1951.

Translated by N.H.C.